Generic Buckling Curves for Specially Orthotropic Rectangular Plates

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Using a double affine transformation, the classical buckling equation for specially orthotropic plates and the corresponding virtual work theorem are presented in a particularly simple fashion. These dual representations are characterized by a single material constant, called the generalized rigidity ratio D^*, whose range is predicted to be the closed interval from 0 to 1 (if this prediction is correct then the numerical results using $D^*>1$ in the specially orthotropic plate literature are incorrect); when natural boundary conditions are considered a generalized Poisson's ratio ϵ is introduced. Thus the buckling results are valid for any specially orthotropic material; hence the curves presented in the text are *generic* rather than specific. The solution trends are twofold; the buckling coefficients decrease with decreasing D^* and, when applicable, they decrease with *increasing* ϵ. Since the isotropic plate is one limiting case of the above analysis, it is also true that isotropic buckling coefficients *decrease* with *increasing* Poisson's ratio.

Nomenclature

a/b	= plate aspect ratio
a_0/b_0	= plate affine aspect ratio
$D_{11},D_{12},D_{66},D_{22}$	= orthotropic plate constants, GPa
D^*	= generalized rigidity ratio $=(D_{12}+2D_{66})/(D_{11}D_{22})^{1/2}$
k_0	= affine plate buckling coefficient $=Pb^2/\pi^2(D_{11}D_{22})^{1/2}$
m	= number of half-waves in the x_0 affine direction if $x=0$, a_0 are simply supported
N_x,N_y,N_{xy}	= plate force resultants, N/m
P	= uniaxial compressive resultant $=-N_x$, N/m
U	= total strain energy, GJ
x,y	= real space plate dimensions, m
x_0,y_0	= affine space plate dimensions, $m^{3/2}\cdot N^{-1/4}$
x, y	$=(D_{11})^{1/4}x_0$, $(D_{22})^{1/4}y_0$, respectively, the double affine transformation, hence, $a/b=(D_{11}/D_{22})^{1/4}(a_0/b_0)$ is the aspect ratio affine transformation
α,β	= roots of the characteristic equation (15), defined by Eqs. (16) and (17)
ϵ	= generalized Poisson's ratio, defined by the identity $D^*=\epsilon D^*+(1-\epsilon)D^*$ and by further defining $\epsilon D^*=D_{12}/(D_{11}D_{22})^{1/2}$ and $(1-\epsilon)D^*=2D_{66}/(D_{11}D_{22})^{1/2}$

I. Introduction

THE known buckling solutions for specially orthotropic rectangular plates are few in number and, as such, provide a very sketchy overview of how the solutions depend on *individual elastic constants*. The failure of these few solutions to provide a comprehensive parametric understanding of the buckling behavior is readily attributed to the existence of four elastic constants, a plate aspect ratio, and (when applicable) indices that specify mode numbers in the x and y directions.

Clearly an idea must be infused into the solution process such that a minimum number of parameters remain. Such a candidate is the affine transformation scheme.[1-4] A cursory review of specially orthotropic plate literature[5-17] reveals‡ that parameters, fully or partially equivalent to the parameters associated with the presently proposed affine transformation process, have been used by numerous investigators for over half a century. Curiously, no one seemed to realize that these parameters are the essential keys for a general description of specially orthotropic plates. As will be seen in the next section, transformations of the form $x=Ax_0$ and $y=By_0$ transform the partial differential equation into an affine space in which the plate affine aspect ratio is a_0/b_0 (the plate aspect ratio in physical space is a/b) and the resulting partial differential equation contains only one material constant, called the generalized rigidity ratio (henceforth denoted by D^*). A secondary material constant ϵ, which is a generalized Poisson's ratio, must be introduced when at least one natural (generalized force) boundary condition is present. Furthermore, as a result of direct calculation for all known specially orthotropic materials, and as a result of a micromechanics formulation for two-phase (fiber-matrix) composite materials,[18] the value of D^* is predicted to be in the closed interval from 0 to 1; that is, $0\le D^*\le 1$. Of the three possible cases discussed in the specially orthotropic plate literature, i.e., case 1: $D^*<1$; case 2: $D^*=1$; case 3: $D^*>1$; usually only case 3 has been used in presenting numerical results.[8-11,13,14] Thus if the above prediction is correct (i.e., $0\le D^*\le 1$) a corollary result of importance is that many numerical results in the orthotropic plate literature are incorrect.§ The general range of ϵ (by direct calculation only) appears to be $0.2\le\epsilon\le 0.4$.

The buckling results are obtained in two distinct categories. In the first category the partial differential equation is used directly to solve for buckling coefficients of uniaxially compressed plates whose compressed sides are simply supported; the boundary conditions of the perpendicular sides are allowed to be five combinations of clamped, simply supported, or free. In the second category the virtual work theorem is used to find pure shear and pure bending buckling coefficients for all sides simply supported via the Ritz-

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‡The authors wish to thank the reviewers for contributions to this list; however, our conclusions are not modified by these inclusions. References 5 and 7 describe (with only partial success) attempts to construct similarity rules. A simple yet complete set of similarity rules has been derived by the first author as outlined in Ref. 4.

§On the other hand, when considering plane stress/plane strain specially orthotropic slab problems, the affine plane stress function equation is $(\partial^4\phi/\partial x_0^4)+2H^*(\partial^4\phi/\partial x_0^2\partial y_0^2)+(\partial^4\phi/\partial y_0^4)=0$, where $H^*=H^*(D^*,\epsilon)$ and $H^*\ge 0$. Therefore it is predicted that most stress function solutions in the specially orthotropic literature are accompanied by *correct* numerical results.

Galerkin technique and the buckling coefficients of a uniaxially compressed plate with all sides clamped is formed via the Budiansky-Hu Lagrangian multiplier method.

II. Governing Equations and Their Affine Counterparts

The buckling equation and its virtual work counterpart are given by (see, e.g., Ref. 14),

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}-\left[N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_y\frac{\partial^2 w}{\partial y^2}\right]=0 \quad (1)$$

and $\delta U=0$, where

$$U=\frac{1}{2}\int_0^a\int_0^b\left\{D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+4D_{66}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2+N_x\left(\frac{\partial w}{\partial x}\right)^2+2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+N_y\left(\frac{\partial w}{\partial y}\right)^2\right\}\mathrm{d}x\mathrm{d}y \quad (2)$$

Introducing $x\equiv(D_{11})^{1/4}x_0$, $y\equiv(D_{22})^{1/4}y_0$ [and hence the plate dimensions a and b transform as $a=(D_{11})^{1/4}a_0$, $b=(D_{22})^{1/4}b_0$ so that the plate aspect ratio transforms as $a/b=(D_{11}/D_{22})^{1/4}(a_0/b_0)$], $D^*\equiv(D_{12}+2D_{66})/(D_{11}D_{22})^{1/2}$, and defining $\epsilon D^*\equiv D_{12}/(D_{11}D_{22})^{1/2}$ so that $(1-\epsilon)D^*\equiv 2D_{66}/(D_{11}D_{22})^{1/2}$ it is seen that Eqs. (1) and (2) become (N_{xy} and N_y are dropped from the partial differential equation formulation)

$$\frac{\partial^4 w}{\partial x_0^4}+2D^*\frac{\partial^4 w}{\partial x_0^2\partial y_0^2}+\frac{\partial^4 w}{\partial y_0^4}+k_0\left(\frac{\pi}{b_0}\right)^2\frac{\partial^2 w}{\partial x_0^2}=0 \quad (3)$$

and

$$U_0=U/(D_{11}D_{22})^{1/4}=\frac{1}{2}\int_0^{a_0}\int_0^{b_0}\left\{\left(\frac{\partial^2 w}{\partial x_0^2}\right)^2+2D^*\left[(1-\epsilon)\left(\frac{\partial^2 w}{\partial x_0\partial y_0}\right)^2+\epsilon\frac{\partial^2 w}{\partial x_0^2}\frac{\partial^2 w}{\partial y_0^2}\right]+\left(\frac{\partial^2 w}{\partial y_0}\right)^2+N_{x_0}\left(\frac{\partial w}{\partial x_0}\right)^2+2N_{x_0y_0}\frac{\partial w}{\partial x_0}\frac{\partial w}{\partial y_0}+N_{y_0}\left(\frac{\partial w}{\partial y_0}\right)^2\right\}\mathrm{d}x_0\mathrm{d}y_0 \quad (4)$$

where the buckling coefficient k_0 has been defined by the relations

$$k_0=Pb^2/\pi^2(D_{11}D_{22})^{1/2}, \qquad P=-N_x \quad (5)$$

and the affine in-plane loads are defined by the relations

$$N_{x_0}=N_x/(D_{11})^{1/2}, \quad N_{x_0y_0}=N_{xy}/(D_{11}D_{22})^{1/4}$$
$$N_{y_0}=N_y/(D_{22})^{1/2} \quad (6)$$

Furthermore, if $w=0$ on the perimeter of the plate, then[19]

$$\int_0^{a_0}\int_0^{b_0}\left\{\frac{\partial^2 w}{\partial x_0^2}\frac{\partial^2 w}{\partial y_0^2}-\left(\frac{\partial^2 w}{\partial x_0\partial y_0}\right)^2\right\}\mathrm{d}x_0\mathrm{d}y_0=0$$

so that Eq. (4) may be cast in the useful but restricted form ($w=0$ on the perimeter) given by

$$U_0=\frac{1}{2}\int_0^{a_0}\int_0^{b_0}\left\{\left(\frac{\partial^2 w}{\partial x_0^2}\right)^2+2D^*\left(\frac{\partial^2 w}{\partial x_0\partial y_0}\right)^2+\left(\frac{\partial^2 w}{\partial y_0}\right)^2+N_{x_0}\left(\frac{\partial w}{\partial x_0}\right)^2+2N_{x_0y_0}\frac{\partial w}{\partial x_0}\frac{\partial w}{\partial y_0}+N_{y_0}\left(\frac{\partial w}{\partial y_0}\right)^2\right\}\mathrm{d}x_0\mathrm{d}y_0 \quad (7)$$

where it is noted that the generalized Poisson's ratio ϵ is now absent from the restricted U_0 expression. Equations (3) and (7) will be used exclusively in the following sections.

When natural boundary conditions are present in a given problem one must have expressions for the moments and the effective shears (units are N and N/m, respectively); thus

$$\begin{Bmatrix} M_x \\ M_y \end{Bmatrix}=\begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22}\end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2\end{Bmatrix} \quad (8a)$$

$$M_{xy}=2D_{66}\frac{\partial^2 w}{\partial x\partial y} \quad (8b)$$

and

$$\begin{Bmatrix} V_x \\ V_y \end{Bmatrix}=\begin{bmatrix} -\dfrac{\partial}{\partial x} & -2\dfrac{\partial}{\partial y} & 0 \\ 0 & -2\dfrac{\partial}{\partial x} & -\dfrac{\partial}{\partial y}\end{bmatrix}\begin{Bmatrix} M_x \\ M_{xy} \\ M_y\end{Bmatrix} \quad (9)$$

are transformed into the affine plane as,

$$\begin{Bmatrix} M_{x_0} \\ M_{y_0} \end{Bmatrix}=\begin{bmatrix} 1 & \epsilon D^* \\ \epsilon D^* & 1\end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x_0^2 \\ \partial^2 w/\partial y_0^2\end{Bmatrix}$$
$$M_{x_0y_0}=(1-\epsilon)D^*\frac{\partial^2 w}{\partial x_0\partial y_0} \quad (10)$$

and

$$V_{x_0}=-\frac{\partial^3 w}{\partial x_0^3}-(2-\epsilon)D^*\frac{\partial^3 w}{\partial x_0\partial y_0^2}$$
$$V_{y_0}=-\frac{\partial^3 w}{\partial y_0^3}-(2-\epsilon)D^*\frac{\partial^3 w}{\partial x_0^2\partial y_0} \quad (11)$$

where the following definitions have been used [Since only homogeneous boundary conditions are used in this paper, these definitions do not assume importance unless one wishes to calculate the moments and shears in the plate (per unit amplitude of the mode shape).]

$$M_{x_0}=M_x/(D_{11})^{1/2}, \quad M_{y_0}=M_y/(D_{22})^{1/2}$$
$$M_{x_0y_0}=M_{xy}/(D_{11}D_{22})^{1/4}$$
$$V_{x_0}=V_x/(D_{11})^{1/4}, \quad V_{y_0}=V_y/(D_{22})^{1/4} \quad (12)$$

III. Buckling Coefficients of Uniaxially Compressed Plates Whose Compressed Sides are Simply Supported

This section reworks the orthotropic counterparts of the isotropic problems first addressed by Timoshenko in 1907[10] (actually the isotropic problem with all four sides simply supported was solved by Bryan[20] in 1891). Now assuming¶ a solution of Eq. (3) to be

$$w(x_0,y_0,D^*,k_0,m) = Y_m(y_0,D^*,k_0)\sin m\pi x_0/a_0 \qquad (13)$$

one finds

$$\frac{d^4 Y_m}{dy_0^4} - 2D^*\left(\frac{m\pi}{a_0}\right)^2 \frac{d^2 Y_m}{dy_0^2} + \left(\frac{m\pi}{a_0}\right)^4 \left[1 - k_0\left(\frac{a_0}{mb_0}\right)^2\right] Y_m = 0 \qquad (14)$$

to be the mode shape equation in the y_0 direction. The roots of the corresponding characteristic equation are given, in several steps, as

$$D^2 = \left(\frac{m\pi}{a_0}\right)^2 \left\{D^* \pm \left[(D^*)^2 - 1 + k_0\left(\frac{a_0}{mb_0}\right)^2\right]^{1/2}\right\} \qquad (15)$$

and if the square root is always positive (and greater than D^*),

$$D = \pm \frac{m\pi}{a_0}\begin{Bmatrix}\alpha^* \\ i\beta^*\end{Bmatrix} = \pm \begin{Bmatrix}\alpha \\ i\beta\end{Bmatrix} \qquad (16)$$

where

$$\begin{Bmatrix}\alpha^* \\ \beta^*\end{Bmatrix} = \left\{\left[(D^*)^2 - 1 + k_0\left(\frac{a_0}{mb_0}\right)^2\right]^{1/2} \pm D^*\right\}^{1/2} \qquad (17)$$

Thus, the general solution for Y_m is given as**

$$Y_m(y_0,D^*,k_0,a_0/b_0) = A_m \sinh\alpha y_0 + B_m \cosh\alpha y_0 + C_m \sin\beta y_0 + D_m \cos\beta y_0 \qquad (18)$$

and for given boundary conditions on $y_0 = 0, b_0$ a 4×4 set of linear homogeneous algebraic equations in A_m, B_m, C_m, and D_m are obtained. For nontrivial solutions the determinant of the coefficient matrix must vanish, thus giving rise to a transcendental equation that provides the solution for k_0 (and the corresponding Y_m). The governing transcendental equations are listed for the various boundary conditions chosen at $y = 0, b_0$ (i.e., the unloaded edges).

Unloaded Edges Simply Supported

One finds $B_m = D_m = 0$ (and eventually $A_m = 0$) and the 4×4 determinant is equal to zero only when $\sin\beta b_0 = 0$. Thus the only closed form solution for k_0 (of all the problems considered) is given by

$$k_0 - 2D^* = \left(\frac{a_0}{mb_0}\right)^2 + \left(\frac{mb_0}{a_0}\right)^2 \qquad (19)$$

One immediately notes that the right-hand side 1) does not contain D^*, and 2) is identical in form to the isotropic buckling coefficient expression in real space. [The expression for the isotropic k when all four sides are simply supported is $k = 2 + (a/mb)^2 + (mb/a)^2 \equiv (a/mb + mb/a)^2$; thus it is noted that $D^* = 1$ (hence $\epsilon = \nu$) corresponds to the isotropic case when, in addition, $D_{11} = D_{22}$. The $D^* = 1$ case has been called the quasi-isotropic case (i.e., $D_{11} \neq D_{22}$ in general) by the present authors.]

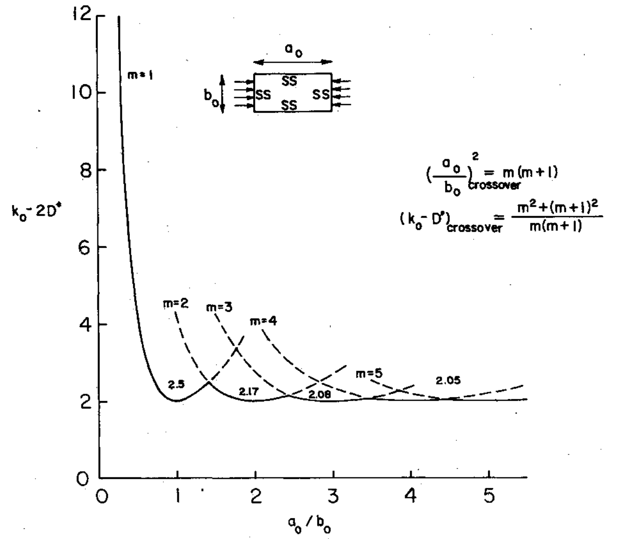

Fig. 1 Uniaxial buckling coefficients (modified) $k_0 - 2D^*$ vs affine aspect ratio a_0/b_0 for SS-SS-SS-SS boundary conditions.

Thus the minima of $k_0 - 2D^*$ are at $a_0/b_0 = m$ ($m = 1,2,3,...$); thus $(k_0)_{min} = 2(D^* + 1)$, and the "crossover points" are given by $a_0/b_0 = [m(m+1)]^{1/2}$ ($m = 1,2,3,...$). Equation (19) is shown in Fig. 1. The graph is used by remembering that

$$\frac{a_0}{b_0} = \left(\frac{D_{22}}{D_{11}}\right)^{1/4} \frac{a}{b}$$

hence use a_0/b_0 to find k_0 from the graph, and $P = (D_{11}D_{22})^{1/2}\pi^2 k_0/b^2$ in units of N/m.

One Unloaded Edge Clamped and the Other Simply Supported

$$\alpha b_0 \sin\beta b_0 - \beta b_0 \tanh\alpha b_0 \cos\beta b_0 = 0 \qquad (20)$$

Equation (20) solved for the minimum k_0 envelopes is shown in Fig. 2, and the minimum value of k_0 is given approximately as

$$(k_0)_{min} = 3.049 + 2.372D^* \qquad 0 \leq D^* \leq 1.0 \qquad (21)$$

Unloaded Edges Clamped

$$2\left[\cos\beta b_0 - \frac{1}{\cosh\alpha b_0}\right] + \left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right)\sin\beta b_0 \tanh\alpha b_0 = 0 \qquad (22)$$

Equation (22) solved for the minimum k_0 envelopes is shown in Fig. 3, and the minimum value of k_0 is given approximately as

$$(k_0)_{min} = 4.516 + 2.46D^* \qquad 0 \leq D^* \leq 1.0 \qquad (23)$$

In these first three cases all of the boundary conditions are geometric; thus the generalized Poisson's ratio ϵ does not enter into the computations for k_0 and the corresponding mode shape Y_m.

One Unloaded Edge Free and the Other Clamped

$$\frac{2}{\cosh\alpha b_0} + \left[\frac{t_1}{t_2} + \frac{t_2}{t_1}\right]\cos\beta b_0 + \left[\frac{\beta t_1}{\alpha t_2} - \frac{\alpha t_2}{\beta t_1}\right]\tanh\alpha b_0 \sin\beta b_0 = 0 \qquad (24)$$

¶This assumption was used by Timoshenko in his 1907 paper and it is thus properly referred to as the Timoshenko method.

**This solution is not valid for SS-F-SS-F boundary conditions. Also note that α and β should be subscripted as α_m and β_m; they are left in the α,β format for ease in writing, reading, and printing.

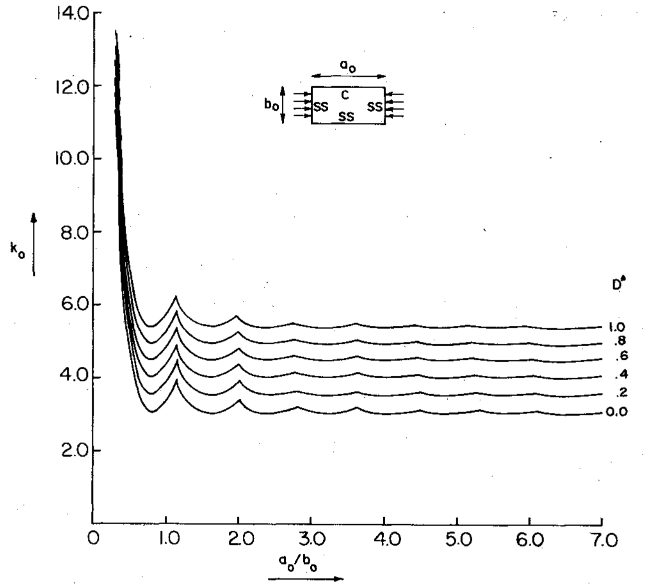

Fig. 2 Uniaxial buckling coefficients k_0 vs affine aspect ratio a_0/b_0 for SS-SS-SS-CL boundary conditions.

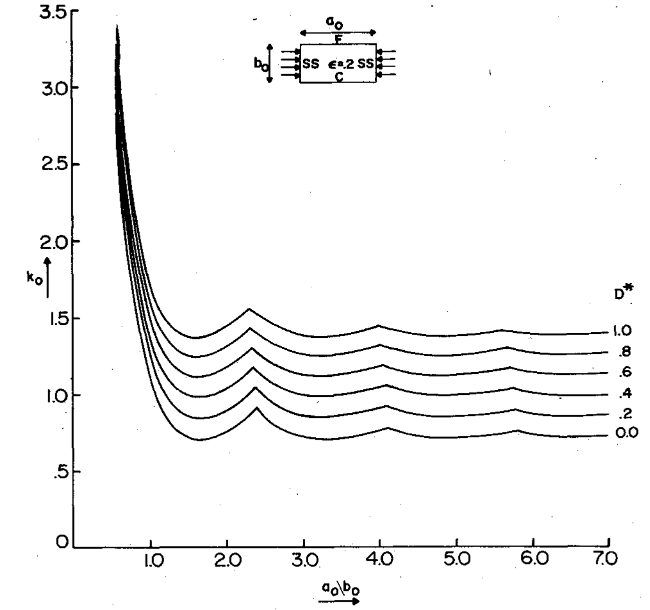

Fig. 4 Uniaxial buckling coefficients k_0 vs affine aspect ratio a_0/b_0 for SS-F-SS-CL boundary conditions ($\epsilon = 0.2$).

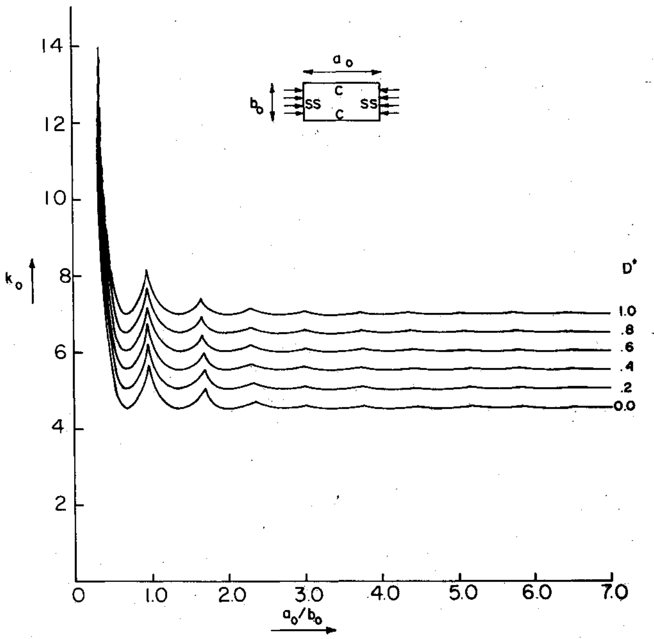

Fig. 3 Uniaxial buckling coefficients k_0 vs affine aspect ratio a_0/b_0 for SS-CL-SS-CL boundary conditions.

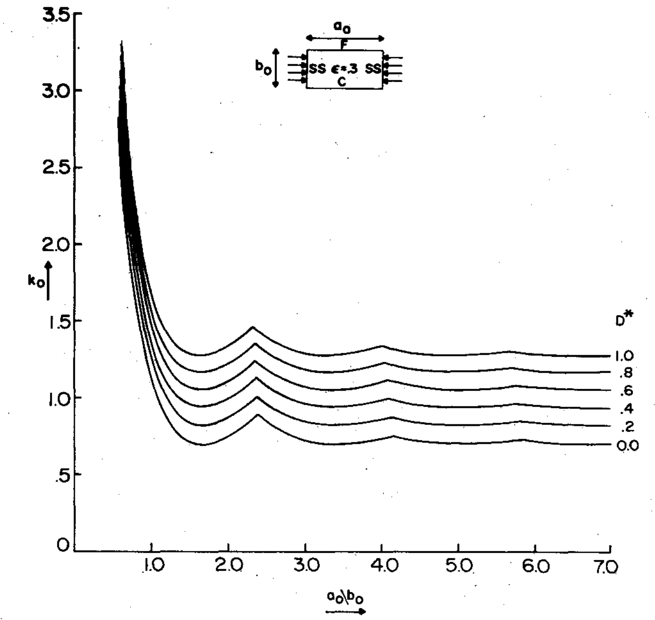

Fig. 5 Uniaxial buckling coefficients k_0 vs affine aspect ratio a_0/b_0 for SS-F-SS-CL boundary conditions ($\epsilon = 0.3$).

where

$$\begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = \begin{bmatrix} 1-\epsilon/2 & \epsilon/2 \\ \epsilon/2 & 1-\epsilon/2 \end{bmatrix} \begin{Bmatrix} (\alpha b_0)^2 \\ (\beta b_0)^2 \end{Bmatrix} \tag{25}$$

Equation (24) solved for the minimum k_0 envelopes is shown in Figs. 4 and 5 for $\epsilon = 0.2$ and 0.3, respectively, and the minimum values are given approximately by

$$(k_0)_{\min} = \begin{Bmatrix} 0.705 + 0.632D^*;\ \epsilon = 0.2 \\ 0.701 + 0.569D^*;\ \epsilon = 0.3 \end{Bmatrix} \qquad 0 \le D^* \le 1 \qquad \begin{matrix} (26) \\ (27) \end{matrix}$$

One Unloaded Edge Free and the Other Simply Supported

$$t_1^2 \tanh \alpha b_0 \cos \beta b_0 - (\alpha/\beta) t_2^2 \sin \beta b_0 = 0 \tag{28}$$

where t_1 and t_2 have been defined by Eq. (25).

Equation (28) solved for the minimum k_0 envelope ($m = 1$) is shown in Figs. 6 and 7 for $\epsilon = 0.2$ and 0.3, respectively, and the asymptotic k_0 values as $a_0/b_0 \to \infty$ are given approximately by

$$(k_0)_{a_0/b_0 \to \infty} = \begin{Bmatrix} 0.476D^*;\ \epsilon = 0.2 \\ 0.420D^*;\ \epsilon = 0.3 \end{Bmatrix} \qquad 0 \le D^* \le 1 \qquad \begin{matrix} (29) \\ (30) \end{matrix}$$

Particularly notice that in these last two cases k_0 *decreased* with *increasing* ϵ. This is a general trend for rectangular geometry and orthotropy. Oyibo[3] has demonstrated the *reverse* general trend for circular geometry and orthotropy.

IV. Various Buckling Coefficients Using the Virtual Work Theorem

In this section *generic results* are presented for three types of loading; the calculations employ the affinely transformed virtual work theorem, Eq. (4), in two variants. The first

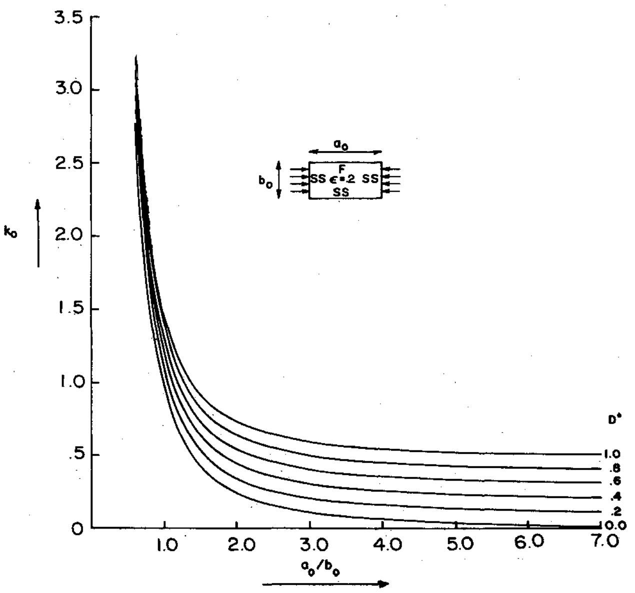

Fig. 6 Uniaxial buckling coefficients k_0 vs affine aspect ratio a_0/b_0 for SS-F-SS-SS boundary conditions ($\epsilon = 0.2$).

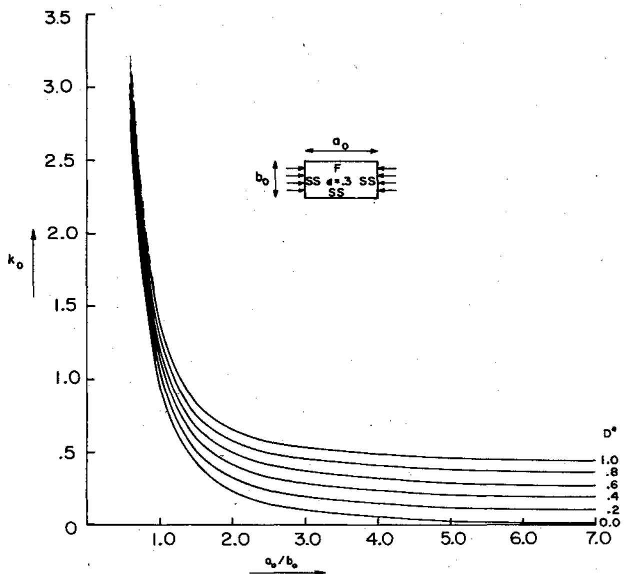

Fig. 7 Uniaxial buckling coefficients k_0 vs affine aspect ratio a_0/b_0 for SS-F-SS-SS boundary conditions ($\epsilon = 0.3$).

variant is the usual Ritz-Galerkin scheme which is used to calculate pure bending, and pure shear buckling coefficients for simply supported plates. The second variant is the Budiansky-Hu[21] Lagrangian multiplier scheme which is used to calculate uniaxial compression buckling coefficients for clamped plates. The quasi-isotropic case ($D^* = 1$ and $\epsilon = \nu$, Poisson's ratio) can be checked with a much more extensive literature since these k_0 vs a_0/b_0 curves are identical to the k vs a/b curves for the isotropic case for corresponding boundary conditions and loadings. The results of this section are presented below.

Pure Bending Buckling Coefficients with All Edges Simply Supported

The results are shown in Fig. 8 and the tabulation of the minimum k_0 values (vs D^*) is displayed in Table 1. An empirical linear curve (for fast reference) with only two-place accuracy which summarizes Table 1 is given by

$$(k_0)_{\min} = 12.87 + 11.03D^*; 0 \leq D^* \leq 1.0 \tag{31}$$

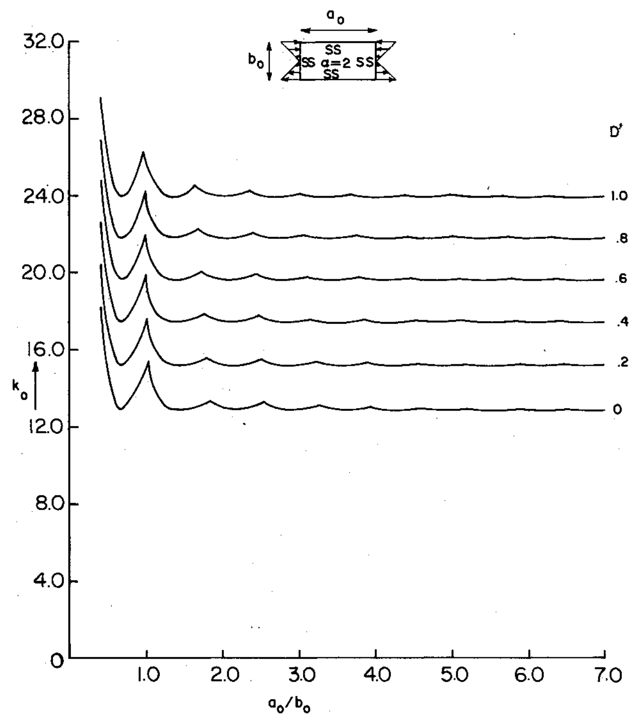

Fig. 8 Pure bending buckling coefficients vs affine aspect ratio a_0/b_0 for SS-SS-SS-SS boundary conditions.

Pure Shear Buckling Coefficients with All Edges Simply Supported

The results are shown in Fig. 9 and the tabulation of the asymptotic k_0 values (vs D^*) is displayed in Table 2. An empirical linear curve which summarizes Table 2 for two-place accuracy is given by

$$(k_0)_{a_0/b_0 \to \infty} = 2.90 + 2.45D^*; 0 \leq D^* \leq 1 \tag{32}$$

Uniaxial Compressive Buckling Coefficients with All Edges Clamped

The results are shown in Fig. 10 and the tabulation of the asymptotic k_0 values (vs D^*) is displayed in Table 3. An empirical linear curve which summarizes Table 3 for two-place accuracy is given by

$$(k_0)_{a_0/b_0 \to \infty} = 4.52 + 2.45D^*; 0 \leq D^* \leq 1.0 \tag{33}$$

V. Discussion

This presentation demonstrates the enormous payoff for recasting the buckling equations (both the partial differential

Table 1 Minimum stability coefficients of a specially orthotropic rectangular plate simply supported on all edges and subjected to pure bending on a pair of those edges

D^*	$k_{0\min}$	D^*	$k_{0\min}$
0	12.8686	0.6	19.5945
0.2	15.1465	0.8	21.7650
0.4	17.3884	1.0	23.9001

Table 2 "Plate strip" stability coefficients of a specially orthotropic rectangular plate simply supported on all edges and subjected to shear stresses on all edges

D^*	$k_{0(a_0/b_0 \to \infty)}$	D^*	$k_{0(a_0/b_0 \to \infty)}$
0	2.9008	0.6	4.3738
0.2	3.3931	0.8	4.8625
0.4	3.8841	1.0	5.3500

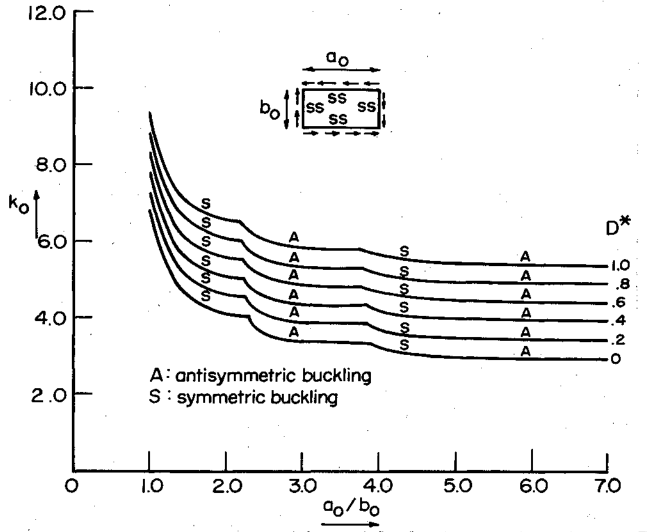

Fig. 9 Pure shear buckling coefficients vs affine aspect ratio a_0/b_0 for SS-SS-SS-SS boundary conditions.

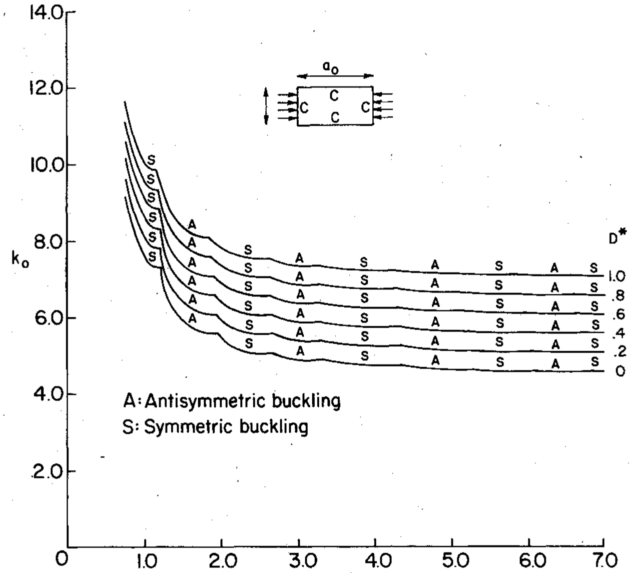

Fig. 10 Uniaxial buckling coefficients vs affine aspect ratio a_0/b_0 for CL-CL-CL-CL boundary conditions.

equation *and* the virtual work theorem) in the affine plane. The payoff is that one truly can do an exhaustive parameter study of buckling solution trends (or vibration solution trends, or static deflection solution trends, i.e., any specially orthotropic plate problem) by just finding the solutions that correspond to $0 \le D^* \le 1$. It is noticed from the problems of the text that, for a given a_0/b_0, the k_0 vs D^* curves vary almost linearly, so that one may accurately interpolate between the D^* values ($D^*=0,0.2,0.4,0.6,0.8,1.0$). Thus these problems never need be recalculated, no matter what new materials become available in the future, so long as the prediction that $0 \le D^* \le 1$ remains valid. As a brief example of solution trends, consider the following example.

Given four specially orthotropic plates all with an aspect ratio $a/b=1$ and all with SS-C-SS-C boundary conditions (hence Fig. 3 is needed), but made of four different materials, determine the buckling coefficients k_0 and the buckling stress σ if thickness to width ratio $t/b=0.01$. The four typical material properties (courtesy of Professor R. B. Pipes, University of Delaware) are for (A) graphite/epoxy, (B) E-glass/epoxy, (C) boron/epoxy, and (D) graphite/aluminum. The material properties are shown in Table 4 and the calculated results (Fig. 3 determines k_0 in terms of a_0/b_0 and D^*) are shown in Table 5 using the relation $\sigma=(t/b)^2(\pi^2/12)\sqrt{Q_{11}Q_{22}}k_0$ where $\sqrt{Q_{11}Q_{22}}=\sqrt{E_1E_2}/[1-\nu_{12}^2(E_2/E_1)]$.

In closing, it is emphasized that $D^*=0$ forms a valid and important lower bound for the solutions, just as $D^*=1$ is an important upper bound, which is called the *quasi-isotropic case* herein. If in addition to $D^*=1$, one allows $D_{11}=D_{22}\rightarrow\mathfrak{D}$ (and hence $\epsilon\rightarrow\nu$ and $a_0/b_0\rightarrow a/b$) then the isotropic plate problem is recovered. [$\mathfrak{D}$ is the familiar flexural rigidity and ν is Poisson's ratio.]

Table 3 "Plate strip" stability coefficients of a specially orthotropic rectangular plate clamped on all edges and compressed on a pair of those edges

D^*	$k_{0(a_0/b_0\rightarrow\infty)}$	D^*	$k_{0(a_0/b_0\rightarrow\infty)}$
0	4.5160	0.6	5.9909
0.2	5.0084	0.8	6.4809
0.4	5.5000	1.0	6.9700

Table 5 Results

Plate	a_0/b_0	D^*	k_0	σ, MPa (psi)
A	0.57	0.36	5.6	20.7 (3000)
B	0.72	0.50	5.9	11.1 (1610)
C	0.51	0.26	5.7	25.7 (3730)
D	0.67	0.92	6.9	32.0 (4650)

Table 4 Material properties of plates

Plate	E_1, GPa (10^6 psi)	E_2, GPa (10^6 psi)	$\sqrt{E_1E_2}$, GPa (10^6 psi)	E_1/E_2[a]	G_{12}/E_2	ν_{12}
A	138 (20)	14.5 (2.1)	44.7 (6.48)	9.52	0.45	0.21
B	42.7 (6.2)	11.7 (1.7)	22.35 (3.25)	3.65	0.35	0.27
C	207 (30)	14.5 (2.1)	54.70 (7.94)	14.29	0.38	0.21
D	124 (18)	24.8 (3.6)	55.45 (8.05)	5.00	0.89	0.30

[a]The $\sqrt{E_1E_2}$ and E_1/E_2 values were calculated from the original psi values and E_1 and E_2; the results were then converted into GPa values.

Acknowledgments

This work was supported by a joint NASA/U.S. Air Force Office of Scientific Research Grant No. NGL 33-018-003; we thank these agencies for their continuing support. It is also a pleasure to thank Ms. Geri Frank and Mrs. Anna Mae Harris for their extraordinary typing skills.

References

[1]Brunelle, E. J., "The Use of a Double Affine Transformation to Obtain a Class of Generalized Elastodynamic Solutions of Orthotropic Plates," Paper presented at the SIAM 1980 National Meeting, June 1980.

[2]Brunelle, E. J., "The Use of Affine Transformations in the Solution of Composite Structures Problems," Paper presented at the Seventeenth Annual Meeting of the Society of Engineering Science, Inc., Dec. 1980.

[3]Oyibo, G. A., "The Use of Affine Transformations in the Analysis of Stability and Vibrations of Orthotropic Plates," Doctoral Thesis, Rensselaer Polytechnic Institute, May 1981.

[4]Brunelle, E. J., "Eigenvalue Similarity Rules for a Class of Rectangular Specially Orthotropic Laminated Plates," Paper presented at the Ninth U.S. National Congress of Applied Mechanics, June 1982.

[5]Wittrick, W. H., "Correlation Between Some Stability Problems for Orthotropic and Isotropic Plates Under Bi-Axial and Uni-Axial Direct Stress," *Aeronautical Quarterly,* Vol. 4, Aug. 1952, pp. 83-92.

[6]Lang, H. A., "The Affine Transformation for Orthotropic Plane-Stress and Plane-Strain Problems," *Journal of Applied Mechanics,* Vol. 23, March 1956, pp. 1-6.

[7]Shulesko, P., "A Reduction Method for Buckling Problems of Orthotropic Plates," *Aeronautical Quarterly,* Vol. 8, May 1957, pp. 145-156.

[8]Lekhnitsky, S. G., *Anisotropic Plates,* translated from 1st Ed., American Iron and Steel Institute, New York, 1956, pp. 134-340 (also published in hard cover by Gordon and Breach, translated from the 2nd Ed., 1968).

[9]Timoshenko, S. P. and Woinowsky-Krieger, S., "Theory of Plates and Shells," 2nd Ed., McGraw-Hill Book Co., New York, 1959, pp. 364-477.

[10]Timoshenko, S. P. and Gere, J. M., *Theory of Elastic Stability,* 2nd Ed., McGraw-Hill Book Co., New York, 1961, pp. 403-408.

[11]Hearmon, R. F. S., *An Introduction to Applied Anisotropic Elasticity,* Oxford University Press, London, 1961, pp. 90-127.

[12]Vinson, J. R. and Brull, M. A., "New Techniques of Solution for Problems in the Theory of Orthotropic Plates," Paper presented at the Fourth U.S. National Congress of Applied Mechanics, June 1962.

[13]Leissa, A. W., "Vibration of Plates," NASA SP-160, 1969, pp. 245-266.

[14]Ashton, J. E. and Whitney, J. M., *Theory of Laminated Plates,* Technomic Press, Stamford, Conn., 1970, pp. 45-76.

[15]Szilard, R., *Theory and Analysis of Plates; Classical and Numerical Methods,* Prentice-Hall, Englewood Cliffs, N.J., 1974, pp. 374-390, 597.

[16]Sakata, T., "Generalized Procedure of the Reduction Method for Vibrating Problems of Orthotropic Plates," *Journal of the Franklin Institute,* Vol. 303, 1977, pp. 415-424.

[17]Bert, C. W., "Fundamental Frequencies of Orthotropic Plates with Various Planforms and Edge Conditions," *Shock and Vibration Bulletin,* Vol. 47, Part 2, Sept. 1977, pp. 89-94.

[18]Brunelle, E. J., Unpublished work; available in summary form in the 40th Semiannual Progress Report, "Composite Structural Materials," Joint NASA-AFOSR Grant No. NGL 33-018-003, July 1981.

[19]Leggert, D. M. A., "The Buckling of a Square Panel under Shear when One Pair of Opposite Edges is Clamped," Aeronautical Research Council, Reports and Memoranda No. 1991, 1941.

[20]Bryan, G. H., "On the Stability of a Plane Plate under Thrusts in Its Own Plane with Application of the Buckling of the Sides of a Ship," *Proceedings of the London Mathematical Society,* Vol. 22, 1891, pp. 54-67.

[21]Budiansky, B. and Hu, P. C., "The Lagrangian Multiplier Method of Finding Upper and Lower Limits to Critical Stresses of Clamped Plates," NACA Rept. 848, 1946.

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